8.4 Properties of Rhombuses, Rectangles, and Squares

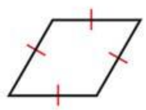
Before You used properties of parallelograms.

Now You will use properties of rhombuses, rectangles, and squares.

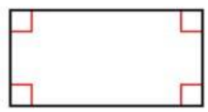
Why? So you can solve a carpentry problem, as in Example 4.

Key Vocabulary
- **rhombus**
- **rectangle**
- **square**

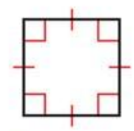
In this lesson, you will learn about three special types of parallelograms: *rhombuses*, *rectangles*, and *squares*.



A rhombus is a parallelogram with four congruent sides.



A rectangle is a parallelogram with four right angles.



A square is a parallelogram with four congruent sides and four right angles.

You can use the corollaries below to prove that a quadrilateral is a rhombus, rectangle, or square, without first proving that the quadrilateral is a parallelogram.

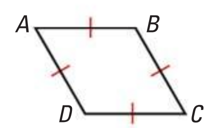
COROLLARIES *For Your Notebook*

RHOMBUS COROLLARY

A quadrilateral is a rhombus if and only if it has four congruent sides.

ABCD is a rhombus if and only if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$.

Proof: Ex. 57, p. 539

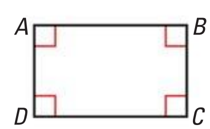


RECTANGLE COROLLARY

A quadrilateral is a rectangle if and only if it has four right angles.

ABCD is a rectangle if and only if $\angle A$, $\angle B$, $\angle C$, and $\angle D$ are right angles.

Proof: Ex. 58, p. 539

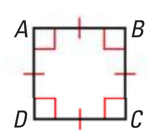


SQUARE COROLLARY

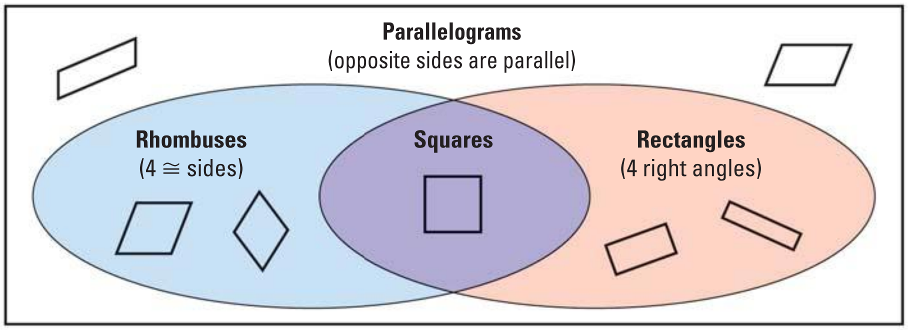
A quadrilateral is a square if and only if it is a rhombus and a rectangle.

ABCD is a square if and only if $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{AD}$ and $\angle A$, $\angle B$, $\angle C$, and $\angle D$ are right angles.

Proof: Ex. 59, p. 539



The *Venn diagram* below illustrates some important relationships among parallelograms, rhombuses, rectangles, and squares. For example, you can see that a square is a rhombus because it is a parallelogram with four congruent sides. Because it has four right angles, a square is also a rectangle.



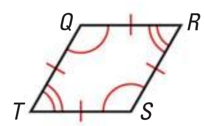
EXAMPLE 1 Use properties of special quadrilaterals

For any rhombus *QRST*, decide whether the statement is *always* or *sometimes* true. Draw a sketch and explain your reasoning.

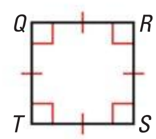
a. $\angle Q \cong \angle S$ **b.** $\angle Q \cong \angle R$

Solution

a. By definition, a rhombus is a parallelogram with four congruent sides. By Theorem 8.4, opposite angles of a parallelogram are congruent. So, $\angle Q \cong \angle S$. The statement is *always* true.

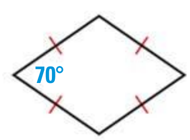


b. If rhombus *QRST* is a square, then all four angles are congruent right angles. So, $\angle Q \cong \angle R$ if *QRST* is a square. Because not all rhombuses are also squares, the statement is *sometimes* true.



EXAMPLE 2 Classify special quadrilaterals

Classify the special quadrilateral. Explain your reasoning.



Solution

The quadrilateral has four congruent sides. One of the angles is not a right angle, so the rhombus is not also a square. By the Rhombus Corollary, the quadrilateral is a rhombus.

GUIDED PRACTICE for Examples 1 and 2

1. For any rectangle *EFGH*, is it *always* or *sometimes* true that $\overline{FG} \cong \overline{GH}$? *Explain* your reasoning.

2. A quadrilateral has four congruent sides and four congruent angles. Sketch the quadrilateral and classify it.

DIAGONALS The theorems below describe some properties of the diagonals of rhombuses and rectangles.

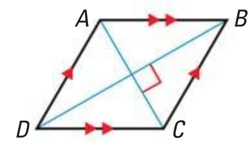
THEOREMS — *For Your Notebook*

THEOREM 8.11

A parallelogram is a rhombus if and only if its diagonals are perpendicular.

$\square ABCD$ is a rhombus if and only if $\overline{AC} \perp \overline{BD}$.

Proof: p. 536; Ex. 56, p. 539

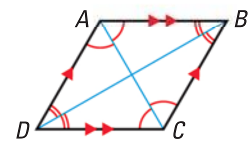


THEOREM 8.12

A parallelogram is a rhombus if and only if each diagonal bisects a pair of opposite angles.

$\square ABCD$ is a rhombus if and only if $\overline{AC}$ bisects $\angle BCD$ and $\angle BAD$ and $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$.

Proof: Exs. 60–61, p. 539

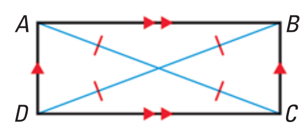


THEOREM 8.13

A parallelogram is a rectangle if and only if its diagonals are congruent.

$\square ABCD$ is a rectangle if and only if $\overline{AC} \cong \overline{BD}$.

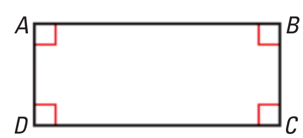
Proof: Exs. 63–64, p. 540



EXAMPLE 3 List properties of special parallelograms

Sketch rectangle *ABCD*. List everything that you know about it.

Solution



By definition, you need to draw a figure with the following properties:

- The figure is a parallelogram.
- The figure has four right angles.

Because *ABCD* is a parallelogram, it also has these properties:

- Opposite sides are parallel and congruent.
- Opposite angles are congruent. Consecutive angles are supplementary.
- Diagonals bisect each other.

By Theorem 8.13, the diagonals of *ABCD* are congruent.

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✓ GUIDED PRACTICE for Example 3

3. Sketch square *PQRS*. List everything you know about the square.

BICONDITIONALS Recall that biconditionals such as Theorem 8.11 can be rewritten as two parts. To prove a biconditional, you must prove both parts.

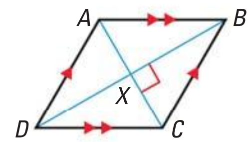
Conditional statement If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

Converse If a parallelogram is a rhombus, then its diagonals are perpendicular.

PROOF Part of Theorem 8.11

PROVE THEOREMS
You will prove the other part of Theorem 8.11 in Exercise 56 on page 539.

If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.



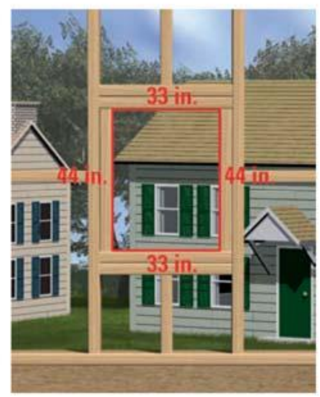
GIVEN ▶ $ABCD$ is a parallelogram; $\overline{AC} \perp \overline{BD}$

PROVE ▶ $ABCD$ is a rhombus.

Proof $ABCD$ is a parallelogram, so $\overline{AC}$ and $\overline{BD}$ bisect each other, and $\overline{BX} \cong \overline{DX}$. Also, $\angle BXC$ and $\angle CXD$ are congruent right angles, and $\overline{CX} \cong \overline{CX}$. So, $\triangle BXC \cong \triangle DXC$ by the SAS Congruence Postulate. Corresponding parts of congruent triangles are congruent, so $\overline{BC} \cong \overline{DC}$. Opposite sides of a $\square ABCD$ are congruent, so $\overline{AD} \cong \overline{BC} \cong \overline{DC} \cong \overline{AB}$. By definition, $ABCD$ is a rhombus.

EXAMPLE 4 Solve a real-world problem

CARPENTRY You are building a frame for a window. The window will be installed in the opening shown in the diagram.



a. The opening must be a rectangle. Given the measurements in the diagram, can you assume that it is? *Explain.*

b. You measure the diagonals of the opening. The diagonals are 54.8 inches and 55.3 inches. What can you conclude about the shape of the opening?

Solution

a. No, you cannot. The boards on opposite sides are the same length, so they form a parallelogram. But you do not know whether the angles are right angles.

b. By Theorem 8.13, the diagonals of a rectangle are congruent. The diagonals of the quadrilateral formed by the boards are not congruent, so the boards do not form a rectangle.

✓ GUIDED PRACTICE for Example 4

4. Suppose you measure only the diagonals of a window opening. If the diagonals have the same measure, can you conclude that the opening is a rectangle? *Explain.*

8.4 EXERCISES

HOMEWORK KEY

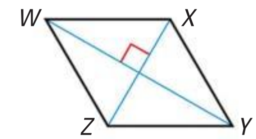
○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 7, 15, and 55

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 30, 31, and 62

SKILL PRACTICE

1. **VOCABULARY** What is another name for an equilateral rectangle?

2. ★ **WRITING** Do you have enough information to identify the figure at the right as a rhombus? *Explain.*



EXAMPLES 1, 2, and 3 on pp. 534–535 for Exs. 3–25

RHOMBUSES For any rhombus *JKLM*, decide whether the statement is *always* or *sometimes* true. Draw a diagram and *explain* your reasoning.

3. $\angle L \cong \angle M$
4. $\angle K \cong \angle M$
5. $\overline{JK} \cong \overline{KL}$
6. $\overline{JM} \cong \overline{KL}$
7. $\overline{JL} \cong \overline{KM}$
8. $\angle JKM \cong \angle LKM$

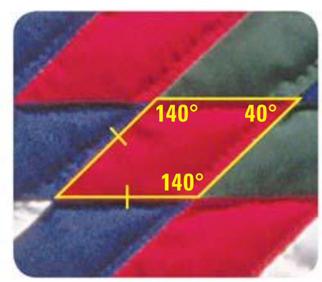
RECTANGLES For any rectangle *WXYZ*, decide whether the statement is *always* or *sometimes* true. Draw a diagram and *explain* your reasoning.

9. $\angle W \cong \angle X$
10. $\overline{WX} \cong \overline{YZ}$
11. $\overline{WX} \cong \overline{XY}$
12. $\overline{WY} \cong \overline{XZ}$
13. $\overline{WY} \perp \overline{XZ}$
14. $\angle WXZ \cong \angle YXZ$

CLASSIFYING Classify the quadrilateral. *Explain* your reasoning.

15. 

16. 

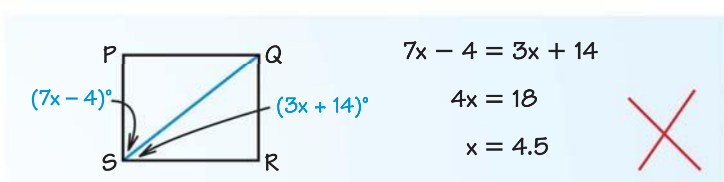
17. 

18. **USING PROPERTIES** Sketch rhombus *STUV*. *Describe* everything you know about the rhombus.

USING PROPERTIES Name each quadrilateral—*parallelogram*, *rectangle*, *rhombus*, and *square*—for which the statement is true.

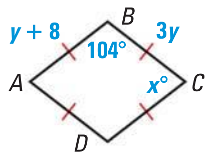
19. It is equiangular.
20. It is equiangular and equilateral.
21. Its diagonals are perpendicular.
22. Opposite sides are congruent.
23. The diagonals bisect each other.
24. The diagonals bisect opposite angles.

25. **ERROR ANALYSIS** Quadrilateral *PQRS* is a rectangle. *Describe* and correct the error made in finding the value of *x*.

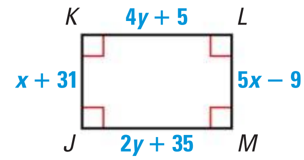


ALGEBRA **Classify the special quadrilateral. *Explain* your reasoning. Then find the values of x and y.**

26.



27.



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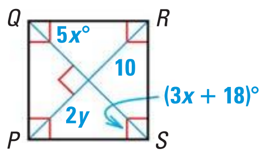

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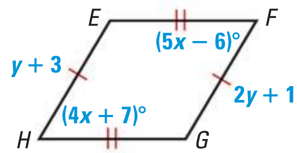
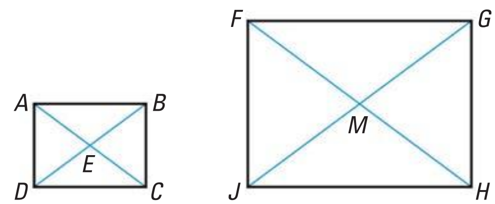


30. ★ **SHORT RESPONSE** The diagonals of a rhombus are 6 inches and 8 inches. What is the perimeter of the rhombus? *Explain*.

31. ★ **MULTIPLE CHOICE** Rectangle *ABCD* is similar to rectangle *FGHJ*. If $AC = 5$, $CD = 4$, and $FM = 5$, what is HJ?

Ⓐ 4 Ⓑ 5

Ⓒ 8 Ⓓ 10



RHOMBUS **The diagonals of rhombus *ABCD* intersect at *E*. Given that $m\angle BAC = 53°$ and $DE = 8$, find the indicated measure.**

32. $m\angle DAC$

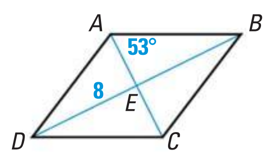
33. $m\angle AED$

34. $m\angle ADC$

35. DB

36. AE

37. AC



RECTANGLE **The diagonals of rectangle *QRST* intersect at *P*. Given that $m\angle PTS = 34°$ and $QS = 10$, find the indicated measure.**

38. $m\angle SRT$

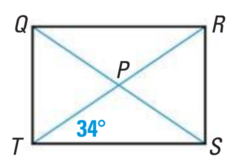
39. $m\angle QPR$

40. QP

41. RP

42. QR

43. RS



SQUARE **The diagonals of square *LMNP* intersect at *K*. Given that $LK = 1$, find the indicated measure.**

44. $m\angle MKN$

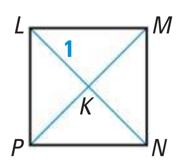
45. $m\angle LMK$

46. $m\angle LPK$

47. KN

48. MP

49. LP



COORDINATE GEOMETRY **Use the given vertices to graph ▱*JKLM*. Classify ▱*JKLM* and *explain* your reasoning. Then find the perimeter of ▱*JKLM*.**

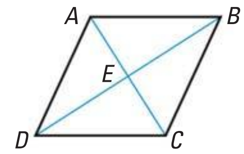
50. $J(-4, 2)$, $K(0, 3)$, $L(1, -1)$, $M(-3, -2)$

51. $J(-2, 7)$, $K(7, 2)$, $L(-2, -3)$, $M(-11, 2)$

○ = WORKED-OUT SOLUTIONS on p. WS1

★ = STANDARDIZED TEST PRACTICE

52. REASONING Are all rhombuses similar? Are all squares similar? *Explain* your reasoning.

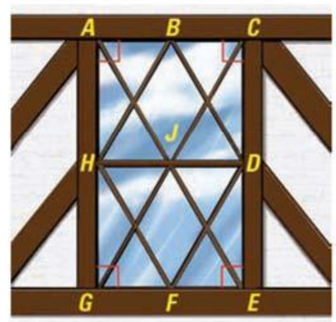


53. CHALLENGE Quadrilateral $ABCD$ shown at the right is a rhombus. Given that $AC = 10$ and $BD = 16$, find all side lengths and angle measures. *Explain* your reasoning.

PROBLEM SOLVING

EXAMPLE 2 on p. 534 for Ex. 54

54. MULTI-STEP PROBLEM In the window shown at the right, $\overline{BD} \cong \overline{DF} \cong \overline{BH} \cong \overline{HF}$. Also, $\angle HAB$, $\angle BCD$, $\angle DEF$, and $\angle FGH$ are right angles.



a. Classify $HBDF$ and $ACEG$. *Explain* your reasoning.

b. What can you conclude about the lengths of the diagonals $\overline{AE}$ and $\overline{GC}$? Given that these diagonals intersect at J, what can you conclude about the lengths of $\overline{AJ}$, $\overline{JE}$, $\overline{CJ}$, and $\overline{JG}$? *Explain*.

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EXAMPLE 4 on p. 536 for Ex. 55

55. PATIO You want to mark off a square region in your yard for a patio. You use a tape measure to mark off a quadrilateral on the ground. Each side of the quadrilateral is 2.5 meters long. *Explain* how you can use the tape measure to make sure that the quadrilateral you drew is a square.

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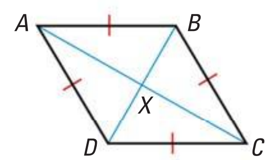
56. PROVING THEOREM 8.11 Use the plan for proof below to write a paragraph proof for the converse statement of Theorem 8.11.



GIVEN ▶ $ABCD$ is a rhombus.

PROVE ▶ $\overline{AC} \perp \overline{BD}$

Plan for Proof Because $ABCD$ is a parallelogram, its diagonals bisect each other at X. Show that $\triangle AXB \cong \triangle CXB$. Then show that $\overline{AC}$ and $\overline{BD}$ intersect to form congruent adjacent angles, $\angle AXB$ and $\angle CXB$.

PROVING COROLLARIES Write the corollary as a conditional statement and its converse. Then *explain* why each statement is true.

57. Rhombus Corollary

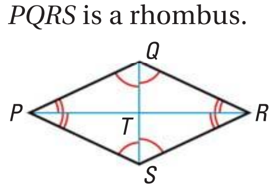
58. Rectangle Corollary

59. Square Corollary

PROVING THEOREM 8.12 In Exercises 60 and 61, prove both parts of Theorem 8.12.

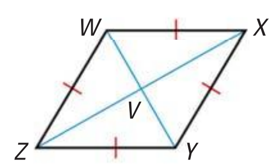
60. GIVEN ▶ $PQRS$ is a parallelogram.
$\overline{PR}$ bisects $\angle SPQ$ and $\angle QRS$.
$\overline{SQ}$ bisects $\angle PSR$ and $\angle RQP$.

PROVE ▶ $PQRS$ is a rhombus.

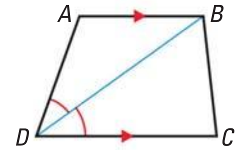


61. GIVEN ▶ $WXYZ$ is a rhombus.

PROVE ▶ $\overline{WY}$ bisects $\angle ZWX$ and $\angle XYZ$.
$\overline{ZX}$ bisects $\angle WZY$ and $\angle YXW$.



62. ★ **EXTENDED RESPONSE** In $ABCD$, $\overline{AB} \parallel \overline{CD}$, and $\overline{DB}$ bisects $\angle ADC$.



a. Show that $\angle ABD \cong \angle CDB$. What can you conclude about $\angle ADB$ and $\angle ABD$? What can you conclude about $\overline{AB}$ and $\overline{AD}$? *Explain.*

b. Suppose you also know that $\overline{AD} \cong \overline{BC}$. Classify $ABCD$. *Explain.*

63. **PROVING THEOREM 8.13** Write a coordinate proof of the following statement, which is part of Theorem 8.13.

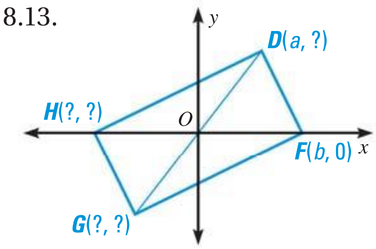
If a quadrilateral is a rectangle, then its diagonals are congruent.

64. **CHALLENGE** Write a coordinate proof of part of Theorem 8.13.

GIVEN ▶ $DFGH$ is a parallelogram, $\overline{DG} \cong \overline{HF}$

PROVE ▶ $DFGH$ is a rectangle.

Plan for Proof Write the coordinates of the vertices in terms of a and b. Find and compare the slopes of the sides.



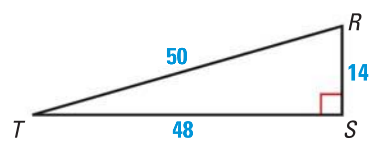
MIXED REVIEW

PREVIEW
Prepare for Lesson 8.5 in Ex. 65.

65. In $\triangle JKL$, $KL = 54.2$ centimeters. Point M is the midpoint of $\overline{JK}$ and N is the midpoint of $\overline{JL}$. Find MN. *(p. 295)*

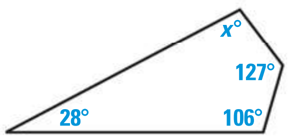
Find the sine and cosine of the indicated angle. Write each answer as a fraction and a decimal. *(p. 473)*

66. $\angle R$ **67.** $\angle T$

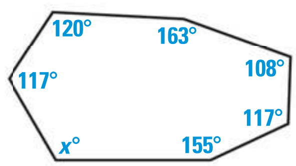


Find the value of x. *(p. 507)*

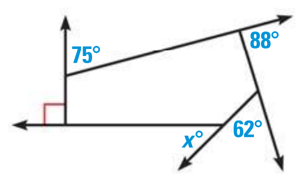
68.



69.



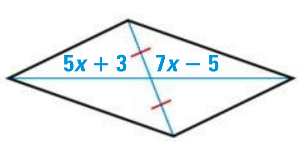
70.



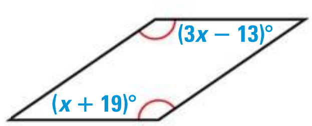
QUIZ *for Lessons 8.3–8.4*

For what value of x is the quadrilateral a parallelogram? *(p. 522)*

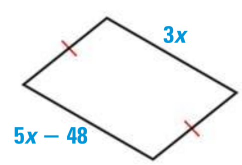
1.



2.

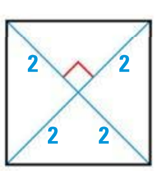


3.

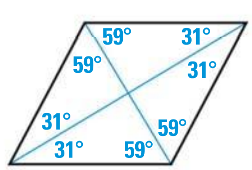


Classify the quadrilateral. ***Explain*** **your reasoning.** *(p. 533)*

4.



5.



6.